An upper and a lower bound for the distance of a manifold to a nearby point

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AN UPPER AND A LOWER BOUND FOR THE DISTANCE OF A MANIFOLD TO A NEARBY POINT

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ABSTRACT

A generalization for underdetermined systems of the wellknown Newton-Kantorovich theorem gives bounds for the distance of a point, say 0, in Hilbert space X to a nearby manifold $S = \{x \in X \mid f(x) = 0\}$. Here $f: X \to Y$ is a differentiable mapping such that $Df(0)$ is surjective; $f(0)$, Df together with right inverse $Df(0)^+$ satisfies typical Kantorovich-like conditions. Analysis in the normal space at 0 of $\tilde{S} = \{x \in X \mid f(x) = f(0)\}$ gives an upperbound of $d(0,S)$. Furthermore the Kantorovich conditions effect S to be locally in a convex cone. The distance of 0 to that cone gives a lowerbound of $d(0,S)$.

1. INTRODUCTION

The purpose of this paper is to derive bounds, both upper bounds as lower bounds, for the distance of a manifold to a nearby point.
The starting point and main tool in the construction of the bounds is the classical Newton-Kantorovich theorem.

THEOREM 0.[2]. Let Y,Z be Banach spaces. Let $B(0,r)$ be an open ball in Banach space Z and let be $\varphi: B(0,r) \subset Z \to Y$ Frechet differentiable on $B(0,r)$ with

$$\|D\varphi(x) - D\varphi(y)\| \leqq L \|x - y\|, \; x,y \in B(0,r). \tag{1.1}$$

Assume that $D\varphi(0)^{-1} \in \mathcal{L}(Z,Y)$ exists,

$$\|D\varphi(0)^{-1}\| \leqq \lambda^{-1}, \; \|D\varphi(0)^{-1} \varphi(0)\| = \tilde{\gamma} \leqq \gamma, \; \kappa := L \gamma \lambda^{-1} < \tfrac{1}{2}$$

and

$$M := \lambda(1-\sqrt{1-2\kappa})/L < r. \tag{1.2}$$

Then the equation $\varphi(x) = 0$ has a solution $z \in B(0,M) \subset Z$ and z is a unique zero of φ in $B(0,\rho_1) \subset Z$, where

$$\rho_1 = \lambda(1+ \sqrt{1-2\kappa})/L$$ □

For the formulation of the distance problem in section two we consider the Hilbert spaces X and Y and we investigate $f: B(0,r) \subset X \to Y$, a Frechet differential mapping. We assume that

(i) $A := Df(0) \in \mathcal{L}(X,Y)$ is surjective and $\|A^+\| \leqq \lambda^{-1}$, where $A^+ \in \mathcal{L}(X,Y)$ is the right inverse of A,

(ii) $$\|Df(x) - Df(y)\| \leqq L \|x-y\|, \; x,y \in B(0,r), \tag{1.4}$$

(iii) $$\|A^+ f(0)\| = \tilde{\gamma} \leq \gamma, \tag{1.5}$$

(iv) $$\kappa = L \gamma \lambda^{-1} < \tfrac{1}{2}, \tag{1.6}$$

and

(v) $M := \lambda(1-\sqrt{1-2\kappa})/L < r.$ (1.7)

We use the following notation:

$$S = \{x \in B(0,r) \mid f(x) = 0\} \tag{1.8}$$

and

$$\tilde{S} = \{x \in B(0,r) \mid f(x) = f(0)\}. \tag{1.9}$$

N_1 denotes Ker(A) being the tangent space of $\tilde{S}$ in 0. Let be N_2 the orthogonal complement of N_1.
Analysis in the normal space N_2 at 0 of $\tilde{S}$ leads to an upperbound of d(0,S), theorem 1. The Kantorovich condition (1.6) effects S to be locally in a convex cone. Theorem 2 gives the distance of 0 to that cone as a lower bound of d(0,S).
This general approach leads to a manageable method to determine sharp error bounds for an approximate solution of an undetermined system.

2. BOUNDS FOR $d(0,S)$

THEOREM 1. The mapping $f\colon B(0,r) \subset X \to Y$ described in the introduction has a zero z in $B(0,M) \cap N_2$; z is the unique zero of $\varphi = f|N_2$ in $B(0,\rho_1) \cap N_2$ where

$$\rho_1 = \lambda(1 + \sqrt{1-2\varkappa})/L. \tag{2.1}$$

PROOF. The surjectivity of A implies that the restriction $A|N_2$ is bijective. Its inverse is also continuous and equals the right inverse $A^+ = A^*(AA^*)^{-1}$ of A [1].
Let be $\varphi = f|N_2$. This mapping $\varphi\colon N_2 \cap B(0,r) \to Y$ satisfies the conditions of the Newton-Kantorovich theorem 0 formulated above. Hence $\varphi(x) = 0$ has a solution z in $B(0,M) \cap N_2$ and z is the unique zero of φ in $B(0,p_1) \cap N_2$. □

LEMMA 1. For the zero z of $\varphi = f|N_2$ in $B(0,M) \cap N_2$ holds

$$\beta := \|z\| \geqq \rho_2 , \tag{2.2}$$

where

$$\rho_2 = \lambda(-1 + \sqrt{1+2\tilde{\varkappa}})/L, \tag{2.3}$$

with $\tilde{\varkappa} = L\tilde{\gamma}\lambda^{-1}$.

PROOF. Since Df is Lipschitz continuous on $B(0,r)$ we have $\|f(z) - f(0) - Az\| \leqq \frac{1}{2} L \beta^2$ and consequently [3]

$$\tilde{\gamma} = \|A^+f(0)\| = \|z + A^+(f(z) - f(0) - Az)\| \leqq \beta + \tfrac{1}{2} L B^2\lambda^{-1},$$

for $z = A^+Az$ as follows from $z \in R(A^*)$.
Hence the positive zero ρ_2 of the quadratic function $t \to \frac{1}{2} L \lambda^{-1} t^2 + t - \tilde{\gamma}$ is majorized by β. That proves (2.3). □

REMARK. As a consequence of (1.7) we get $\beta < M < 2\gamma$. Hence

$$L\beta < 2L\gamma < \lambda. \ \square \tag{2.4}$$

In the sequel we use the following notation

$$V = \{x \in X \,|\, |x| < \beta\},\ P(x) := Df(x)|N_1,\ Q(x) := Df(x)|N_2,\ x \in V, \tag{2.5}$$

$$\alpha = \frac{L\ \beta}{\lambda - L\ \beta}, \tag{2.6}$$

where, as above $\beta = |z|$.

LEMMA 2. $Q(x)$ is regular for $x \in V$ and

$$|Q(x)^{-1}\ P(x)| \leqq \alpha, \quad x \in V. \tag{2.7}$$

PROOF. Let be $x \in V$ and $y = y_1 + y_2$, $y_i \in N_i$, $i = 1,2$. Then $Df(x)y = P(x)y_1 + Q(x)y_2$. Since $P(0) = 0$, $|Px| \leq |Df(x) - Df(0)| \leqq L\beta$, $x \in V$. Similarly $|Q(x) - Q(0)| \leqq |Df(x) - Df(0)| \leqq L\beta$, $x \in V$. Since $Q(0) = D\varphi(0)$,

$$|(Q(x) - Q(0))\ Q(0)^{-1}| \leqq L\ \beta\lambda^{-1} < 1$$

as follows from $\beta < M$ and (2.4). This implies that

$$Q(x) = (I + (Q(x) - Q(0))\ Q(0)^{-1})Q(0)$$

is invertible for each $x \in V$ and

$$|Q(x)^{-1}| \leqq |Q(0)^{-1}|\ (1 - L\beta\ \lambda^{-1})^{-1} \leqq (\lambda - L\beta)^{-1}.$$

Hence $|Q(x)^{-1}\ P(x)| \leqq L\beta(\lambda - L\beta)^{-1} = \alpha$. □

For reasons of shortness we define

$$W = \{x = x_1 + x_2 \in X \,|\, \alpha\ |x_1| + |x_2| < \beta, x_i \in N_i,\ i = 1,2\} \tag{2.8}$$

and with $w = w_1 + w_2 \in X$, $w_i \in N_i$, $i = 1,2$ (2.9)

$$K(w) = \{(1-\tau)w_1 + x_2 \,|\, \tau \in [0,1],\ x_2 \in N_2,\ |x_2 - w_2| \leqq \alpha\ |w_1|\ \tau\}.$$

The lines along which the proof of theorem 2 will be given can be explained with a figure.

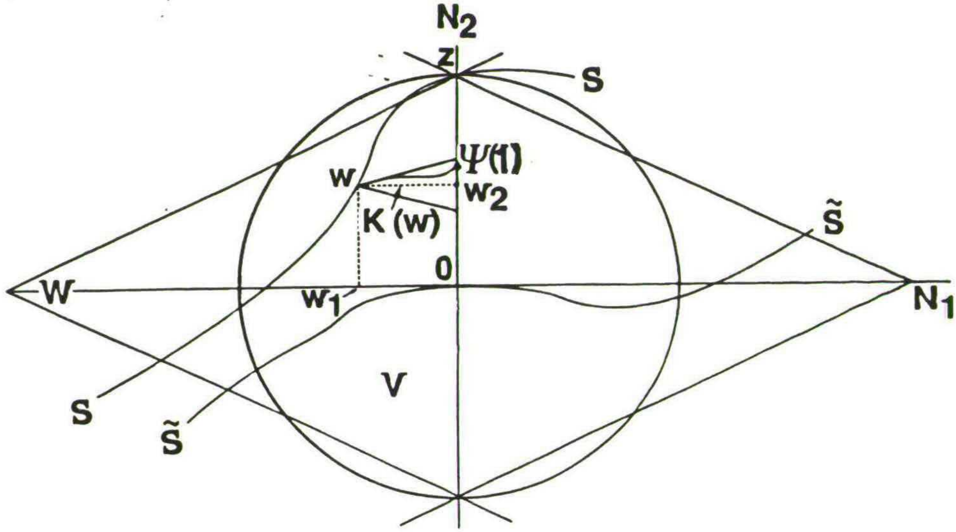

fig. 1. $w \in S \cap W \cap V$ contradicts $S \cap V \cap N_2 = \emptyset$.

In lemma 3 we prove that $w \in W \cap V$ implies $K(w) \subset W \cap V$. In lemma 4 indirectly we prove that $w \in W \cap V$ implies $f(w) \neq 0$. So $S \subset V^c \cup W^c$ and $d(0,S) \geqq d(0,W^c)$. In this manner we get $m := d(0,W^c)$ as an upper bound for the distance of 0 to the manifold S.

LEMMA 3. If $w = w_1 + w_2 \in W \cap V$ then $K(w) \subset V \cap W$.
PROOF. Let be $x = x_1 + x_2 \in K(w)$. Then there exist ε, $\tau \in [0,1]$ and a unit vector $v \in N_2$ such that $x_1 = (1-\tau)w_1$ and $x_2 = w_2 + \varepsilon\alpha\,|w_1|\,\tau v$. Thus

$$|x|^2 = (1-\tau)^2|w_1|^2 + |w_2 + \varepsilon\alpha|w_1|\tau v|^2 \leqq (1-\tau)^2|w_1|^2 + (|w_2| + \alpha\tau|w_1|)^2 = g(\tau).$$

Now $g(0) = |w|^2 < \beta^2$ and $g(1) = (|w_2| + \alpha|w_1|)^2 < \beta^2$ for $w \in V$ and $w \in W$ respectively. So $x \in V$.
Similarly we have $\alpha|x_1| + |x_2| \leqq \alpha|w_1|(1-(1-\varepsilon)\tau) + |w_2| < \beta$. Thus also $x \in W$. □

LEMMA 4. The function f has no zero in $W \cap V$.

PROOF. Assume $w = w_1 + w_2 \in W \cap V$, $w_i \in N_i$, $i = 1,2$ and $f(w) = 0$.
Define a function G as follows

$$(\tau,x_2) \to G(\tau,x_2) = f((1-\tau)w_1+x_2)\ ,\ x_2 \in N_2,\ (1-\tau)^2\ |w_1|^2 + |x_2|^2 < r^2.$$

Then $G(0,w_2) = 0$ and the derivative $D_2G(0,w_2)$ of G in $(0,w_2)$ with respect to x_2 equals $Q(w)$. By lemma two $Q(w)$ is regular. According to the implicit function theorem there exists a $\delta>0$ and a differentiable function $\psi:(-\delta,\delta)\to N_2 \cap V$ such that $\psi(0) = w_2$ and for $\tau \in (-\delta,\delta)$ holds

$$G(\tau,\psi(\tau)) = 0, \; D\psi(\tau) = -D_2G(\tau,\psi(\tau))^{-1} D_1G(\tau,\psi(\tau))$$

where D_1 and D_2 denote differentiation with respect to τ and x_2 respectively. Since

$$D_1G(\tau,x_2) = -P((1-\tau)w_1+x_2)w_1, \; D_2G(\tau,x_2) = Q((1-\tau)w_1+x_2)$$

we have

$$\| D\psi(\tau)\| \leqq \| Q((1-\tau)w_1 + \psi(\tau))^{-1} P((1-\tau)w_1 + \psi(\tau))\| \, \|w_1\| \leqq \alpha \|w_1\|, \; |\tau| < \delta,$$

as follows from lemma 2. Consequently

$$\|\psi(\tau) - w_2\| = \| \int_0^\tau D\psi(\sigma)d\sigma\| \leqq \alpha \|w_1\|\tau, \; 0 < \tau < \delta.$$

Hence $(1-\tau)w_1 + \psi(\tau) \in K(w)$ if $\tau \in [0,\delta) \subset [0,1]$. If $\tau_1, \tau_2 \in [0,\delta)$ then $\|\psi(\tau_1) - \psi(\tau_2)\| \leqq \alpha \|w_1\| \, |\tau_1-\tau_2|$ which implies, by the Cauchy criterion, that $\tilde{w} = \lim_{\tau\uparrow\delta}((1-\tau)w_1+\psi(\tau))$ exists in the closed $K(w)$ and thus $\tilde{w} \in V \cap W$ as follows from lemma 3. So the function ψ can be prolonged and extended until τ equals 1, i.e. $G(1,\psi(1)) = 0$. Thus $f(\psi(1) = 0$. That means $\varphi(\psi(1)) = 0$, with $\psi(1) \in V \cap N_2$ and $\varphi = f|N_2$. This contradicts theorem 1. Hence $w \in W \cap V$ implies $f(w) \neq 0$. □

THEOREM 2. Let $f: B(0,r) \subset X \to Y$ satisfy the conditions given in the introduction and let be

$$g(\tau):= \tau(\lambda/L-\tau)(\tau^2+(\lambda/L-\tau)^2)^{-\frac{1}{2}} \quad , \; 0 < \tau < \frac{\lambda}{L} . \tag{2.10}$$

Then

$$d(0,S) \geqq m:= \begin{cases} g(M) & ,\frac{1}{4}\sqrt{3} \leqq \kappa < \frac{1}{2} \text{ and } \sqrt{1-2} + \frac{1}{2}(1-2\kappa) \leqq \tilde{\kappa} \leqq \kappa \\ g(\rho_2) & ,0 < \kappa < \frac{1}{2} \text{ and } \tilde{\kappa} \leqq \min\{\kappa,\sqrt{1-2\kappa}+\frac{1}{2}(1-2\kappa)\} \end{cases} \tag{2.11}$$

where M and ρ_2 as given in (1.7) and (2.3) respectively.

PROOF. With simple computations we find

$$d(0,W^c) = \beta(1+\alpha^2)^{-\frac{1}{2}} = g(\beta) < \beta,$$

where α and β are given in (2.6) and (2.2) respectively. So

$$d(0,S) \geqq d(0,W^c \cup V^c) = g(\beta).$$

It is easy to see that $\tau = \frac{1}{2}\lambda/L$ is the axis of symmetry of the graph of g. The function g increases on $(0, \frac{1}{2}\lambda/L]$ from 0 until its maximum $\frac{1}{4}\lambda\sqrt{2}/L$ and decreases on $[\frac{1}{2}\lambda/L, \lambda/L)$ to zero. Since $\rho_2 \leqq \beta \leqq M$ as we know from theorem 1 and lemma 1,

$$d(0,S) \geqq m := \min\{g(\rho_2), g(M)\}.$$

The symmetry of g implies that

$$m = g(M) \Leftrightarrow M \geqq \tfrac{1}{2}\lambda/L \text{ and } \rho_2 \geqq \lambda/L - M \tag{2.12}$$

and

$$m = g(\rho_2) \Leftrightarrow M < \tfrac{1}{2}\lambda/L \text{ or } (M \geqq \tfrac{1}{2}\lambda/L \text{ and } \rho_2 < \lambda/L-M). \tag{2.13}$$

With (1.7) we find $M \geqq \frac{1}{2}\lambda/L$ iff $\kappa \geqq \frac{3}{8}$ and with (2.3) we get that $\rho_2 \geqq \lambda/L - M$ iff $\sqrt{1-2\kappa} + \frac{1}{2}(1-2\kappa) \leqq \tilde{\kappa} \leqq k$. Since $\kappa \geq \sqrt{1-2\kappa} + \frac{1}{2}(1-\kappa)$ for $\kappa \geq \frac{1}{4}\sqrt{3}$, (2.11) can be concluded. □

COROLLARY. If $\gamma = \tilde{\gamma}$, i.e. $\kappa = \tilde{\kappa}$, then

$$m = d(0,S) \geqq \begin{cases} g(M) & , \frac{1}{4}\sqrt{3} \leqq \kappa < \frac{1}{2} \\ g(\rho_2) & , \kappa < \frac{1}{4}\sqrt{3} \end{cases} \tag{2.14}$$

PROOF. The two conditions (2.12) and (2.14) lead to the two cases of (2.14) with the same means as in the theorem. □

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